THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050 Mathematical Analysis (Spring 2018)
Tutorial on Apr 11

If you find any mistakes or typos, please email them to ypyang@math.cuhk.edu.hk

Part I: Some comments.

- In Section 5.3 we establish some **very important** properties of continuous functions on an interval. These results may seem to be "obvious", but you should pay attention to the hypotheses of these theorems **(5.3.2/4/5/7/9/10)** and try to show by constructing counterexamples that if any one of the hypotheses is dropped, then the statement will not be true anymore. Therefore, you must study examples accompanying the theorems to gain a better understanding.

- **(Image of an interval)** You should be able to distinguish Theorem 5.3.9 and 5.3.10 and try to construct examples indicating that the continuous image of an open interval (a, b) can be any type of interval, i.e., not necessarily an open interval or a closed interval.

- There are other statements of the **Location of Roots Theorem (or Bolzano's Theorem)**: if f is continuous on $[a, b]$, then

$$f(a)f(b)<0 \implies \exists c \in (a, b) \text{ such that } f(c) = 0,$$
$$f(a)f(b)\leq 0 \implies \exists c \in [a, b] \text{ such that } f(c) = 0.$$

  In application, these two statements are usually more convenient . Please verify them yourself.

- I should not emphasize the importance of this section any more and I expect you to familiarize yourself with all materials and exercises of Section 5.3. In Part III, I will also select some additional exercises that are also important and you may have a look during the reading week.

Part II: Exercises from the textbook.

1. **(Ex 5.3.11)** Let $I := [a, b]$ and $f : I \to \mathbb{R}$ be continuous on I, and assume that $f(a) < 0, f(b) > 0$. Let $W := \{x \in I : f(x) < 0\}$ and let $w := \sup W$. Prove that $f(w) = 0$. (This provides an alternative proof of Theorem 5.3.5)

   Proof: $a \in W \implies W$ is not empty. $W \subset I = [a, b] \implies W$ is bounded. Therefore, $w = \sup W$ exists and $w \in (a, b)$.

   Case 1. If $f(w) < 0$, then $w \in W$. Recall the **Order preservation property** and there exists $\delta > 0$ such that $f(x) < 0, \forall x \in V_\delta(w) \subset I$. Therefore, $w + \delta \in W$, which is a contradiction.

   Case 2. If $f(w) > 0$, similarly we have $f(x) > 0, \forall x \in V_\delta(w) \subset I$ for some $\delta > 0$. Thus $w - \delta$ is also an upper bound for W, also a contradiction.

   Therefore, we conclude that $f(w) = 0$.

2. **(Ex 5.3.13)** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $\lim_{x\to-\infty} f = 0$ and $\lim_{x\to\infty} f = 0$. Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or minimum on $\mathbb{R}$. Give an example to show that both a maximum and a minimum need not be attained.

   Remark: You can consider the same question if the domain of f is replaced by a bounded open interval (a, b). You may also think about what happens if $\lim_{x\to\infty} f(x) \neq \lim_{x\to-\infty} f(x)$.

Part III: Additional exercises.

3. Let $f : [a, b] \to [a, b]$ be continuous. Show that f has a fixed point. (A point c is said to be a **fixed point** of $f(x)$ if $f(c) = c$)

   Proof: Notice that $f(a) \geq a, f(b) \leq b$. Define $F(x) = f(x) - x$, then F is continuous on $[a, b]$ with $F(a)F(b) \leq 0$. By Bolzano's Theorem, there exists $c \in [a, b]$ such that $F(c) = 0$, which means $f(c) = c$ and therefore f has a fixed point c.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$.

   (a) Show that f attains its minimum on $\mathbb{R}$.

   Remark: The result can be also generalized to a continuous function f defined on a bounded open interval (a, b).

   (b) **(Optional)** If we assume in addition that the minimum value is attained at some point $a \in \mathbb{R}$ and $f(a) < a$. Show that there are at least two different points where the function $f(f(x))$ attains its minimum value.

   Proof: **(a)** Because $\lim_{x\to-\infty} f(x) = \infty$, there exists $M > 0$ such that if $x < -M$ then $f(x) > f(0)$. Similarly, there exists $K > 0$ such that if $x > K$ then $f(x) > f(0)$.

   Since f is continuous on the closed interval $[-M, K]$, there exists some $x_0 \in [-M, K]$ such that $f(x_0) \leq f(x), \forall x \in [-M, K]$. Especially we have $f(x_0) \leq f(0)$ and consequently $f(x_0) < f(x)$ for all $x < -M$ or $x > K$. Therefore, $f(x_0) \leq f(x), \forall x \in \mathbb{R}$ and f attains its minimum on $\mathbb{R}$.

   (b) Let $F(x) = f(x) - a$, then $F(a) < 0$ and $\lim_{x\to\infty} F(x) = \lim_{x\to-\infty} F(x) = \infty$. From the Intermediate Value Theorem, there exist $a_1 < a < a_2$ such that $F(a_1) = F(a_2) = 0$.

   Therefore, $f(a_1) = f(a_2) = a$ and $f(f(a_1)) = f(f(a_2)) = f(a)$.

5. (a) **(Generalization of Ex 5.3.6)** Suppose $f : [0, 2a] \to \mathbb{R}$ is continuous and $f(0) = f(2a)$. Show that there exists some point $c \in [0, a]$ such that $f(c) = f(c + a)$.

   (b) Let $f : [0, 1] \to [0, \infty)$ be continuous and $f(0) = f(1) = 0$. Show that for any $r \in (0, 1)$, there exists $c \in [0, 1]$ such that $c + r \in [0, 1]$ and

   $$f(c) = f(c + r).$$

   Remark: The condition $f(0) = f(1) = 0$ cannot be weakened as $f(0) = f(1)$. Also, the condition that f is nonnegative cannot be dropped. Try to find a counterexample for each case.

(c) Let $f : [0,1] \to \mathbb{R}$ be continuous and $f(0) = f(1)$. Show that for any $n \in \mathbb{N}$, there exists $x_n \in \left[0, 1 - \frac{1}{n}\right]$ such that

$$f\left(x_n + \frac{1}{n}\right) = f(x_n).$$

6. Let $f_n(x) = x^n + x$.

(a) Show that for any $n > 1$, the equation $f_n(x) = 1$ has a unique root in the interval $\left(\frac{1}{2}, 1\right)$.

(b) Denote this root by c_n. Show that $\lim_{n\to\infty} c_n$ exists and evaluate it.

Remark: You may practice the same exercise with $f_n(x)$ replaced by $P_n(x) = x + x^2 + \cdots + x^n$, where the sequence of roots (c_n) will converge to another number $\left(\text{which is } \frac{1}{2}\right)$.

7. Let $f : [a,b] \to \mathbb{R}$ be continuous and suppose f has a unique maximum point x_0 in $[a,b]$. If $(x_n) \subset [a,b]$ is a sequence such that $\lim_{n\to\infty} f(x_n) = f(x_0)$. Show that $\lim_{n\to\infty} x_n = x_0$.

Proof: Otherwise (by Theorem 3.4.4), there exists $\varepsilon_0 > 0$ and a subsequence (x_{n_k}) such that

$$|x_{n_k} - x_0| \geq \varepsilon_0, \ \forall k \in \mathbb{N}.$$

Since $(x_{n_k}) \subset [a,b]$ is bounded, from Bolzano-Weierstrass Theorem, it has a convergent subsequence $(x_{n_{k_j}})$. Suppose $\lim_{j\to\infty} x_{n_{k_j}} = x_1$, then $x_1 \in [a,b]$ and $|x_1 - x_0| \geq \varepsilon_0$ from the Order-Preservation Theorem. In addition, $\lim_{j\to\infty} f(x_{n_{k_j}}) = f(x_1)$ because f is continuous.

Notice that $(f(x_{n_{k_j}}))$ is a subsequence of $(f(x_n))$ and thus $\lim_{j\to\infty} f(x_{n_{k_j}}) = f(x_0)$. Therefore, $f(x_1) = f(x_0)$ and x_1 is another maximum point and contradiction arises. So we conclude that $\lim_{n\to\infty} x_n = x_0$.